Tutorial: Relations, Functions, and Equivalent sets

1. Let R be the raltion $R = \{(a,b) \,|\, a < b\}$ on the set of integers $\mathbb{Z}$. Find
   (a) R^{-1} (b) R^c.

2. How many relations are there on a set with n elements that are
   (a) symmetric (b) antisymmetric (c) reflexive.

3. Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$.

4. Let A be the set of non-zero integers and let R be the relation on $A \times A$ defined as follows: $(a,b)\, R\, (c,d)$ whenever $ad = bc$. Prove that R is an equivalence relation.

5. Consider the set of integers $\mathbb{Z}$. Define aRb if $b = a^r$ for some positive integer r. Show that R is a partial on $\mathbb{Z}$.

6. Give an example of relations R on $A = \{1,2,3\}$ having the following property.
   (a) R is both symmetric and antisymmetric (b) R is neither symetric not antisymmetric.
   (c) R is transitive but $R \cup R^{-1}$ is not transitive.

7. Let R be the following equivalence relation on the set $A = \{1, 2, \ldots, 6\}$:

   $R = \{(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)\}$. Find the equivalence classes of R, i.e., find the partition of A induced by R.

8. Let $f : A \to B$ be a function and $E, F \subseteq A$ and $G, H \subseteq B$. Then show that
   (a) $f(E \cup F) = f(E) \cup f(F)$ (b) $f(E \cap F) \subseteq f(E) \cap f(F)$
   (c) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ (d) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.

9. (a) Show that if $f : A \to B$ is injective and $E \subseteq A$, then $f^{-1}(f(E)) = E$. Give an example that equality need not hold if f is not injective.

   (b) Show that if $f : A \to B$ is surjective and $H \subseteq B$, then $f(f^{-1}(H)) = H$. Give an example that equality need not hold if f is not surjective.

10. Let A and B be sets with $|A| = l$ and $|B| = m$.

    (a) Find the number of injective functions from A to B.

    (b) Find the number of surjective functions from A to B.

    (c) Find the number of bijective functions from A to B.

11. Show that $(0, \infty) \approx (-\infty, \infty) \approx (-\frac{\pi}{2}, \frac{\pi}{2})$.

12. $(0,1) \times (0,1) \approx (0,1)$.

13. $\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$.

14. $[0,1] \approx \mathcal{P}(\mathbb{N})$ (Power set of $\mathbb{N}$).

15. Show that $\mathcal{P} = \{p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \,:\, a_0, a_1, \ldots, a_n \in \mathbb{Z}\} \approx \mathbb{N}$.

16. A real number r is called algebraic is r is a solution of $p(x) = 0$, where $p(x) \in \mathcal{P}$ (in above). Show that the set A of all algebraic number is equivalent to $\mathbb{N}$.